IMM-NYU 337
FEBRUARY 1965

NEW YORK UNIVERSITY
COURANT INSTITUTE OF
MATHEMATICAL SCIENCES

# An Integral Equation from Diffraction Theory

A. S. PETERS

PREPARED UNDER
CONTRACT NO. NONR-285(55)
WITH THE
OFFICE OF NAVAL RESEARCH
NR 062-160

NR 062-160

IMM-NYU 337
February 1965

# AN INTEGRAL EQUATION FROM DIFFRACTION THEORY

A. S. Peters

This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, with the Office of Naval Research, Contract No. Nonr-285(55).

c.2

AN INTEGRAL EQUATION FROM
DIFFRACTION THEORY

A. S. Peters

## 1. Introduction

The main purpose of this report is to show how to solve the singular integral equation

$$(1.1) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{i\beta(x-t)}}{x-t} \right\} dt = h(x) + \lambda\Phi(x) \qquad 0 < x$$

where $k$ is an arbitrary constant; $\alpha$ is real; $\beta = \pm \alpha$ ; and the second integral means the Cauchy principal value. Particular cases of this equation, namely

$$(1.2) \qquad \int_0^\infty \Phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} \pm \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) \qquad 0 < x ,$$

were introduced and solved by D. S. Jones [1], [2]; in connection with a problem in diffraction theory. In [2], Jones shows that if any one of (1.2) is written

$$\int_0^\infty \phi(t)K_j(x,t)dt = h(x) \tag{1.3}$$

then there exists a resolvent kernel $R_j(x,y)$ such that

$$\int_0^\infty R_j(\xi,x)K_j(x,t)dx = \delta(t-\xi) ;$$

and it follows from this that

$$\phi(\xi) = \int_0^\infty R_j(\xi,x)h(x)dx . \tag{1.4}$$

In the proof of (1.4) Jones starts with the exhibition of the function $R_j(x,y)$ . The analysis which leads to the choice of $R_j(x,y)$ in the first place is not presented. In a second paper [3] on (1.2), Jones shows that this equation can be reduced to a pair of dual integral equations; which he solves in terms of elementary functions. The solution, however, involves triple integrals and the subsequent analysis which is used in [3] to return to (1.4) is not inconsiderable. In Section 2 we show a more simple analysis of the dual equations.

The dual equations technique is not appropriate for the analysis of

$$(1.5) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} \pm \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad 0 < x .$$

However, in Section 3 we show that the transform

$$\Phi(u) = \int_0^\infty e^{iut} \phi(t)dt$$

satisfies a Cauchy singular integral equation of standard type. In Section 4 we point out that (1.5) is a particular case of the more general equation

$$(1.6) \qquad \int_0^\infty K(x-t)\phi(t)dt + \int_0^\infty K_1(x+t)\phi(t)dt = h(x) + \lambda\phi(x)$$

and we show how the solution of this equation can be reduced to the solution of a Hilbert-Riemann problem when $K_1(T)$ is related to $K(T)$ in a special way.

The methods of Sections 3 and 4 are combined in Sections 5 and 6 to show how the solutions of

$$(1.7) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{-i\alpha(x-t)}}{x-t} \right\} \phi(t)dt = h(x) + \lambda\phi(x)$$

and

$$(1.8) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \frac{ke^{i\alpha(x-t)}}{x-t} \right\} \phi(t)dt = h(x) + \lambda\phi(x)$$

can be found.

## 2. The Reduction to Dual Equations.

Let us start with

$$(2.1)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{-i\alpha(t-x)}}{t-x}+\frac{e^{-i\alpha(t+x)}}{t+x}\right\}dt = h(x) \qquad x > 0$$

where $\alpha$ is real and positive. We suppose that $\phi(t)$ satisfies a uniform Hölder condition when $t$ is positive and finite. This guarantees the existence of the Cauchy principal value of the first integral for each positive and finite value of $x$. Equation (2.1) is the same as

$$(2.2)\qquad \int_0^\infty \phi_1(t)\left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{-i(t+x)}}{t+x}\right\}dt = h_1(x) \qquad x > 0$$

where

$$\phi_1(t) = \phi\left(\frac{t}{\alpha}\right)\ ; \quad h_1(x) = h\left(\frac{x}{\alpha}\right) .$$

Jones remarked that

$$\int_0^\infty \left\{\frac{e^{-i(t-x)}}{t-x}+\frac{e^{i(t+x)}}{t+x}\right\}\cos\lambda x\,dx = \begin{cases} -\pi\, i\cos\lambda t & 0 < \lambda < 1 \\ \pi\sin\lambda t & 1 < \lambda \end{cases}$$

and using this, he reduced (2.2) to the pair of dual equations

$$(2.3)\qquad \int_0^\infty \phi_1(t)\cos\lambda t\,dt = \frac{i}{\pi}\int_0^\infty h_1(x)\cos\lambda x\,dx \qquad 0 < \lambda < 1$$

$$(2.4) \qquad \int_0^\infty \phi_1(t) \sin \lambda t \, dt = \frac{1}{\pi}\int_0^\infty h_1(x) \cos \lambda x \, dx \qquad 1 < \lambda .$$

In order to solve these, Jones introduced $g_1(\lambda)$ such that

$$(2.5) \qquad \int_0^\infty \phi_1(t) \sin \lambda t \, dt = g_1(\lambda) \qquad 0 < \lambda < 1 .$$

This, with (2.4), implies

$$(2.6) \qquad \frac{\pi}{2} \phi_1(t) = \int_0^1 g_1(\lambda) \sin t\lambda d\lambda + \frac{1}{\pi}\int_1^\infty \sin t\lambda \int_0^\infty h_1(x)\cos \lambda x dx d\lambda$$

After the substitution of (2.6) in (2.3), the use of generalized functions leads to a Cauchy type integral equation which is not difficult to solve for $g_1(\lambda)$ ; and when $g_1(\lambda)$ is known (2.6) gives $\phi_1(t)$ . As was noted in the introduction, the representation for $g_1(t)$ that is obtained in this way is rather complicated, and it seems to require intricate analysis to pass from the form (2.6) to the simpler form (1.4).

Instead of the above method let us follow a method that was developed in [4] by the author. First notice that if $\lambda = 0$ the terms of (2.3) may be infinite. In order to cover this possibility we introduce the generalized function $\delta(\lambda)$ and write (2.3) and (2.4) as

$$(2.7) \qquad \int_0^\infty \phi_1(t) \cos \lambda t dt = \frac{1}{\pi}\int_0^\infty h_1(x) \cos \lambda x dx + c_1\delta(\lambda) \quad 0 \le \lambda < 1$$

$$(2.8)\qquad \int_0^\infty \phi_1(t)\sin \lambda t dt = \frac{1}{\pi}\int_0^\infty h_1(x)\cos \lambda x dx \qquad 1 < \lambda .$$

Now if we multiply each side of (2.7) by $2/\pi\,\sqrt{r^2 - \lambda^2}$ ; integrate with respect to $\lambda$ from 0 to $r < 1$ ; and use

$$J_0\,(rt) = \frac{2}{\pi}\int_0^r \frac{\cos t\lambda d\lambda}{\sqrt{r^2-\lambda^2}}$$

we find that (2.7) becomes

$$(2.9)\qquad \int_0^\infty \phi_1(t)\, J_0(rt)dt = \frac{i}{\pi}\int_0^\infty h_1(x)\, J_0(rx)dx + \frac{C}{r} \qquad 0 < r < 1$$

In a similar way, if we use

$$J_0(rt) = \frac{2}{\pi}\int_r^\infty \frac{\sin t\lambda d\lambda}{\sqrt{\lambda^2-r^2}}$$

and

$$Y_0(rx) = -\frac{2}{\pi}\int_r^\infty \frac{\cos x\lambda d\lambda}{\sqrt{\lambda^2-r^2}}$$

we find that (2.8) becomes

$$(2.10)\qquad \int_0^\infty \phi_1(t)\, J_0(rt)dt = -\frac{1}{\pi}\int_0^\infty h_1(x)Y_0(rx)dx \qquad 1 < r .$$

The results (2.9) and (2.10) show that the zeroth order Bessel transform of $\phi_1(t)$ is

$$\int_0^\infty \phi_1(t)J_0(rt)dt = \begin{cases} \frac{i}{\pi}\int_0^\infty h_1(x)J_0(rx)dx + \frac{C}{r} & 0 < r < 1 \\ \\ \frac{1}{\pi}\int_0^\infty h_1(x)Y_0(rx)dx & 1 < r . \end{cases}$$

The inversion of this transform produces

$$(2.11) \quad \frac{\phi_1(t)}{t} = \frac{i}{\pi}\int_0^1 rJ_0(tr)\int_0^\infty h_1(x)J_0(rx)dxdr + C\int_0^1 J_0(tr)dr$$

$$- \frac{1}{\pi}\int_1^\infty rJ_0(tr)\int_0^\infty h_1(x)Y_0(rx)dxdr$$

for the solution of (2.2).

The representation (2.11) can be simplified by using the formula

$$(2.12) \quad \int^z zC_\mu(kz)C_\mu^*(\ell z)dz = z\frac{[kC_{\mu+1}(kz)C_\mu^*(\ell z) - \ell C_\mu(kz)C_{\mu+1}^*(\ell z)]}{k^2 - \ell^2}$$

in which $C_\mu(T)$ and $C_\mu^*(T)$ are any two cylindrical functions of the same order. [See Watson, Theory of Bessel Functions.] Let us interpret the third integral in (2.11) as

$$-\frac{1}{\pi}\lim_{T \to t + i0}\int_1^\infty \mathcal{R} H_0^{(1)}(Tr)\int_0^\infty h_1(x)Y_0(rx)dxdr$$

where $\mathcal{R} H_0^{(1)}(Tr)$ denotes the real part of the Hankel function. Then the use of (2.12) gives

$$\frac{\phi_1(t)}{t} = \frac{i}{\pi}\int_0^\infty h_1(x)\,\frac{[tJ_1(t)J_0(x)-xJ_0(t)J_1(x)]}{t^2-x^2}\,dx + C\int_0^1 J_0(tr)dr$$

$$+ \frac{1}{\pi}\int_0^\infty h_1(x)\,\frac{[tJ_1(t)Y_0(x)-xJ_0(t)Y_1(x)]}{t^2-x^2}\,dx$$

which is the same as

$$(2.13)\qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\,\frac{[txJ_0(t)H_1^{(2)}(x)-t^2J_1(t)H_0^{(2)}(x)]}{x^2-t^2}\,dx$$

$$+ Ct\int_0^1 J_0(tr)dr.$$

If $C = 0$ , (2.13) gives

$$\phi(t) = \phi_1(\alpha t) = \frac{i\alpha}{\pi}\int_0^\infty h(x)\,\frac{[txJ_0(\alpha t)H_1^{(2)}(\alpha x)-t^2J_1(\alpha t)H_0^{(2)}(\alpha x)]}{x^2-t^2}\,dx\ .$$

This solution of (2.1) was presented by Jones in [1], [2]. It shows the meaning of one of the resolvent kernels noted in the introduction, namely,

$$R_1(x,t) = \frac{i\alpha}{\pi}\ \frac{[txJ_0(\alpha t)H_1^{(2)}(\alpha x)-t^2J_1(\alpha t)H_0^{(2)}(\alpha x)]}{x^2-t^2}\ .$$

If we refer to (2.12) again we see that (2.13) can also be

written as

$$(2.14) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x) \left\{ t\int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2i}{\pi}\cdot\frac{t}{x^2-t^2} \right\} dx$$

$$+ C\int_0^t J_0(u)du .$$

The result (2.13) was given by Jones in [3]. It must be noted, however, that $\int_0^t J_0(u)du$ satisfies the homogeneous dual equations only when these equations are interpreted from the point of view of summability, or generalized functions, that is to say, if $v = \lambda + i\varepsilon$, then

$$\lim_{\varepsilon \to 0+} \mathcal{R} \int_0^\infty e^{ivt} \int_0^t J_0(u)dudt = 0 \qquad 0 \leq \lambda < 1$$

$$\lim_{\varepsilon \to 0+} \mathcal{I} \int_0^\infty e^{ivt} \int_0^t J_0(u)dudt = 0 \qquad 1 < \lambda$$

but

$$\int_0^\infty \int_0^t J_0(u)du \cos \lambda tdt$$

and

$$\int_0^\infty \int_0^t J_0(u)du \sin \lambda tdt$$

do not exist in the ordinary sense. This implies that $\int_0^t J_0(u)du$

does not satisfy the homogeneous equation corresponding to (2.2), namely,

$$\int_0^\infty \phi_0(t) \left\{ \frac{e^{-i(t-x)}}{t-x} + \frac{e^{-i(t+x)}}{t+x} \right\} dt = 0 . \tag{2.15}$$

In fact, it can be verified that

$$\int_0^\infty \int_0^t J_0(u)du \left\{ \frac{e^{-i(t-x)}}{t-x} + \frac{e^{-i(t+x)}}{t+x} \right\} dt = -i\pi .$$

Furthermore, it is not difficult to show that if we set $h_1(x) = i\pi$ in

$$\phi_1(t) = \frac{i}{\pi} \int_0^\infty h_1(x) \left\{ t \int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2i}{\pi} \cdot \frac{t}{x^2-t^2} \right\} dx ,$$

then

$$\phi_1(t) = \int_0^t J_0(T)dT .$$

The equation

$$\int_0^\infty \phi(t) \left\{ \frac{e^{-i\alpha(t-x)}}{t-x} - \frac{e^{-i\alpha(t+x)}}{t+x} \right\} dt = h(x) \quad x > 0 , \tag{2.16}$$

where $\alpha$ is real and positive, can be reduced to a pair of dual integral equations by taking the sine transform. This equation is the same as

$$(2.17) \qquad \int_0^\infty \phi_1(t) \left\{ \frac{e^{-i(t-x)}}{t-x} - \frac{e^{-i(t+x)}}{t+x} \right\} dt = h_1(x) \qquad x > 0$$

and if we note that

$$\int_0^\infty \left\{ \frac{e^{-i(t-x)}}{t-x} - \frac{e^{-i(t+x)}}{t+x} \right\} \sin \lambda x dx = \begin{cases} -\pi i \sin \lambda t & 0<\lambda<1 \\ -\pi \cos \lambda t & 1<\lambda \end{cases}$$

we can see that the sine transform of (2.17) yields the equations

$$(2.18) \qquad \int_0^\infty \phi_1(t) \sin \lambda t dt = \frac{i}{\pi} \int_0^\infty h_1(x) \sin \lambda x dx \qquad 0 < \lambda < 1$$

$$(2.19) \qquad \int_0^\infty \phi_1(t) \cos \lambda t dt = - \frac{1}{\pi} \int_0^\infty h_1(x) \sin \lambda x dx \qquad 1 < \lambda \ .$$

If we regard these equations from the point of view of the theory of generalized functions, then differentiation with respect to $\lambda$ changes them into

$$(2.20) \qquad \int_0^\infty t\phi_1(t) \cos \lambda t dt = \frac{i}{\pi} \int_0^\infty h_1(x) x \cos \lambda x dx + c_2 \delta(\lambda-1) \qquad 0<\lambda<1$$

$$(2.21) \qquad \int_0^\infty t\phi_1(t) \sin \lambda t dt = \frac{1}{\pi} \int_0^\infty h_1(x) x \cos \lambda x dx + c_2 \delta(\lambda-1) \qquad 1 \leq \lambda \ .$$

The function $\delta(\lambda-1)$ is introduced because the equations (2.18) and (2.19) show a jump discontinuity at $\lambda = 1$ . The last equa-

tions have the same form as (2.7) and (2.8); and hence if we operate on (2.20) and (2.21) as we did on the pair (2.7), (2.8) we find

$$\int_0^\infty t\phi_1(t)J_0(rt)dt = \begin{cases} \frac{i}{\pi}\int_0^\infty xh_1(x)J_0(rx)dx + c_2\delta(r-1) & 0<r\leq 1 \\ -\frac{1}{\pi}\int_0^\infty xh_1(x)Y_0(rx)dx + c_2\delta(r-1) & 1\leq r. \end{cases}$$

The inversion of this yields

$$(2.22) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^1 rJ_0(tr)\int_0^\infty xh_1(x)J_0(rx)dxdr + c_2J_0(t)$$

$$- \frac{1}{\pi}\int_1^\infty rJ_0(tr)\int_0^\infty xh_1(x)Y_0(rx)dxdr$$

for the solution of (2.17). If we integrate in the same way as we did above we get

$$\phi_1(t) = \frac{i}{\pi}\int_0^\infty xh_1(x)\,\frac{[tJ_1(t)J_0(x) - xJ_0(t)J_1(x)]}{t^2 - x^2}\,dx + c_2J_0(t)$$

$$+ \frac{1}{\pi}\int_0^\infty xh_1(x)\,\frac{[tJ_1(t)Y_0(x) - xJ_0(t)Y_1(x)]}{t^2 - x^2}\,dx$$

or

$$(2.23) \qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\,\frac{[x^2J_0(t)H_1^{(2)}(x) - xtJ_1(t)H_0^{(2)}(x)]}{x^2 - t^2}\,dx$$

$$+ c_2J_0(t) .$$

This can also be written as

$$(2.24)\qquad \phi_1(t) = \frac{i}{\pi}\int_0^\infty h_1(x)\left\{x\int_0^1 uH_0^{(2)}(xu)J_0(tu)du + \frac{2ix}{\pi(x^2-t^2)}\right\}dx$$

$$+ c_2\, J_0(t).$$

The solution (2.24) indicates that $J_0(t)$ may satisfy

$$(2.25)\qquad \int_0^\infty \phi_0(t)\left\{\frac{\epsilon^{-i(t-x)}}{t-x} - \frac{\epsilon^{-i(t+x)}}{t+x}\right\}dt = 0 .$$

This is actually the case. In contrast with what we found above for $\int_0^t J_0(u)du$ and the homogeneous equation (2.15), it can be verified that $J_0(t)$ satisfies (2.25) and the homogenous dual equations corresponding to (2.18), (2.19); without invoking summability procedures.

## 3. Reduction to a Cauchy Integral Equation.

We turn here to the equation

$$(3.1)\qquad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x)+\lambda\phi(x) \quad 0 < x$$

where $\alpha$ is real; $\nu$ is plus or minus one; and $\phi(t)$ is subject to the uniform Hölder condition prescribed in Section 2. We proceed to give a technique which reduces (3.1) to a simple, well-known integral equation of Cauchy type.

We assume that the right hand Fourier transform

$$\Phi(\zeta) = \int_0^\infty e^{i\zeta t}\, \phi(t)dt \qquad\qquad \zeta = \xi + i\eta$$

exists almost everywhere for $\mathscr{I}\zeta = 0$. The Hölder continuity assumption is sufficient for the validity of

$$(3.2)\qquad \phi(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itu}\, \Phi(u)du \quad .$$

These assumptions imply that $\Phi(\zeta)$ can have no poles on the real axis. Furthermore, $\Phi(\zeta)$ is analytic for $\mathscr{I}\zeta > 0$ and $\Phi(\zeta) \to 0$ when $|\zeta| \to \infty$ with $0 < \arg \zeta < \pi$. The substitution of (3.2) in (3.1) gives

$$\frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-itu}\, \Phi(u)du \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x)+\lambda\phi(x)$$

This is the same as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\underline{\Phi}(u)\left\{-e^{ixu}\int_{-\infty}^{\infty}\frac{e^{i(u-\alpha)T}}{T}\,dT+\nu e^{-ixu}\int_{-\infty}^{\infty}\frac{e^{i(u-\alpha)}}{T}\,dT\right\}du$$

$$= h(x) + \lambda\phi(x)$$

or

$$(3.3)\qquad \frac{1}{2}\int_{-\infty}^{\infty}\underline{\Phi}(u)[-ie^{ixu}+i\nu e^{-ixu}]\,\mathrm{sgn}(u-\alpha)du = h(x)+\lambda\phi(x)$$

where

$$\text{signum}\ \xi = \mathrm{sgn}\ \xi = \begin{cases} 1 & \xi > 0 \\ -1 & \xi < 0 \end{cases}.$$

Note that the transformation can be reversed from (3.3) to (3.1).

The result of multiplying (3.3) by $e^{i\zeta x}$, $\mathcal{I}\zeta > 0$, and integrating from zero to infinity is

$$(3.4)\qquad \frac{1}{2}\int_{-\infty}^{\infty}\frac{\underline{\Phi}(u)\ \mathrm{sgn}\ (u-\alpha)du}{u+\zeta}+\frac{\nu}{2}\int_{-\infty}^{\infty}\frac{\underline{\Phi}(u)\ \mathrm{sgn}\ (u-\alpha)du}{u+\zeta}$$

$$= H(\zeta) + \lambda\,\underline{\Phi}(\zeta) \qquad\qquad \mathcal{I}\zeta > 0$$

where

$$H(\zeta) = \int_0^{\infty} e^{i\zeta x}h(x)dx\quad .$$

From here on to the end of this section we suppose for convenience that $\alpha > 0$ . If we let $L$ denote the path along the real axis from $-\infty$ to $\alpha$ ; and let $M$ denote the path along the real axis from $\alpha$ to $+\infty$ we see from (3.4) that

$$- \frac{1}{2} \int_L \frac{\Phi(u)du}{u+\zeta} + \frac{1}{2} \int_M \frac{\Phi(u)du}{u+\zeta} - \frac{\nu}{2} \int_L \frac{\Phi(u)du}{u-\zeta} + \frac{\nu}{2} \int_M \frac{\Phi(u)du}{u-\zeta}$$

$$= H(\zeta) + \lambda \Phi(\zeta) .$$

Let $L$ be hinged at $\alpha$ and then rotated through the upper half plane into coincidence with $M$ . The result of doing this is

$$\int_M \Phi(u) \left[\frac{1}{u+\zeta} + \frac{\nu}{u-\zeta}\right] du - \nu\pi i \, \Phi(\zeta) = H(\zeta) + \lambda \, \Phi(\zeta) .$$

We suppose now that each of $H(\xi)$ and $\Phi(\xi)$ is continuous for $\xi > \alpha$ . This insures the existence of

$$\lim_{\substack{\zeta \to \xi \\ \xi \to \alpha}} \int_L \Phi(u) \left[\frac{1}{u+\zeta} + \frac{\nu}{u-\zeta}\right] du$$

and it follows that

$$(3.5) \qquad \int_\alpha^\infty \Phi(u) \left[\frac{1}{u+\xi} + \frac{\nu}{u-\xi}\right] du = H(\xi) + \lambda \, \Phi(\xi) \qquad \alpha < \xi .$$

This equation reduces to well known equations if $\nu$ is either $1$ or $-1$ .

If $\nu = 1$ we have

$$(3.6) \qquad \int_{\alpha}^{\infty} \frac{\Phi(u)2udu}{u^2-\xi^2} = H(\xi) + \lambda\,\Phi(\xi)$$

or, after an obvious transformation,

$$\int_0^{\infty} \frac{\Phi(\sqrt{V+\alpha^2})dV}{V-T} = H(\sqrt{T+\alpha^2}) + \lambda\,\Phi(\sqrt{T+\alpha^2}) \ .$$

The last equation can be solved in various ways. The solution is well known and for $\lambda \neq \pm \pi i$ it is

$$\Phi(\sqrt{V+\alpha^2}) = -\frac{\lambda H(\sqrt{V+\alpha^2})}{\lambda^2+\pi^2} - \frac{V^{1-\theta}}{(\lambda^2+\pi^2)} \int_0^{\infty} \frac{H(\sqrt{T+\alpha^2})dT}{T^{1-\theta}(T-V)} + \frac{C}{V^{\theta}}$$

where $\theta$ is defined by

$$\pi \cot \pi\theta = \lambda \qquad\qquad 0 \leq \theta < 1 \ .$$

Hence

$$(3.7) \qquad \Phi(u) = \frac{-\lambda H(u)}{\lambda^2+\pi^2} - \frac{2(u^2-\alpha^2)^{1-\theta}}{(\lambda^2+\pi^2)} \int_{\alpha}^{\infty} \frac{\xi H(\xi)d\xi}{(\xi^2-\alpha^2)^{1-\theta}(\xi^2-u^2)}$$

$$+ \frac{c}{(u^2-\alpha^2)}\,\theta \ .$$

Similarly if $\nu = -1$ we have

$$(3.8) \qquad 2\int_{\alpha}^{\infty} \frac{\overline{\Phi}(u)du}{u^2-\xi^2} = -\frac{H(\xi)}{\xi} - \frac{\lambda\overline{\Phi}(\xi)}{\xi}$$

and if $\lambda \neq \pm \pi i$ the solution of this is

$$(3.9) \qquad \overline{\Phi}(u) = \frac{-\lambda H(u)}{\lambda^2+\pi^2} + \frac{2u}{(\lambda^2+\pi^2)(u^2-\alpha^2)^w}\int_{\alpha}^{\infty} \frac{(\xi^2-\alpha^2)^w H(\xi)d\xi}{\xi^2-u^2}$$

$$+ \frac{cu}{(u^2-\alpha^2)^w}$$

where $w$ is defined by

$$\pi \cot \pi w = -\lambda \qquad 0 \leq w < 1 \quad .$$

Let us apply the above method to the equation

$$(3.10) \qquad \int_0^{\infty} \phi(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = h(x) \quad .$$

[This is the equation we obtain if we replace $\alpha$ in (2.16) by $-|\alpha|$ and $h(x)$ by $-h(x)$ .] According to what we have above, the right hand Fourier transform of $\phi(t)$ must satisfy

$$\int_{\alpha}^{\infty} \frac{\overline{\Phi}(u)\cdot 2udu}{u^2-\xi^2} = H(\xi) \quad .$$

As we can see from (3.7), with $\lambda = 0$ ,

$$(3.11)\qquad \Phi(u) = \frac{-2\sqrt{u^2-\alpha^2}}{\pi^2}\int_\alpha^\infty \frac{\xi H(\xi)d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} + \frac{c}{\sqrt{u^2-\alpha^2}}$$

$$= -\frac{2}{\pi^2}\int_0^\infty h(x)\sqrt{u^2-\alpha^2}\int_\alpha^\infty \frac{\xi e^{i\xi x}d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)}dx + \frac{c}{\sqrt{u^2-\alpha^2}}\ .$$

In order to show that the inverse of the right hand side of (3.11) can be expressed in a form similar to that shown in (2.24) we will use the facts that

$$\frac{d}{d\xi}\int_0^\alpha \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{\xi\sqrt{u^2-\alpha^2}}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} + \frac{u}{u^2-\xi^2}$$

and

$$\frac{d}{d\xi}\int_0^\xi \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{u}{u^2-\xi^2}\ .$$

Then

$$\sqrt{u^2-\alpha^2}\int_\alpha^\infty \frac{\xi e^{ix\xi}d\xi}{\sqrt{\xi^2-\alpha^2}\,(\xi^2-u^2)} = \int_\alpha^\infty e^{ix\xi}\frac{d}{d\xi}\int_0^\alpha \frac{\lambda d\lambda\ d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+\int_\alpha^\infty \frac{ue^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -ix\int_0^\alpha \frac{\lambda}{\sqrt{u^2-\lambda^2}}\int_\lambda^\infty \frac{e^{ix\xi}d\xi d\lambda}{\sqrt{\xi^2-\lambda^2}} - \int_0^\alpha e^{ix\xi}\frac{d}{d\xi}\int_0^\xi \frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+\int_\alpha^\infty \frac{ue^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= \frac{\pi x}{2} \int_0^\alpha \frac{\lambda H_0^{(1)}(\lambda x)\,d\lambda}{\sqrt{u^2-\lambda^2}} + \int_0^\infty \frac{e^{ix\xi}u\,d\xi}{\xi^2-u^2}$$

$$= -\frac{\pi i x}{2}\int_0^\infty e^{iut}\int_0^\alpha \lambda J_0(\lambda t)H_0^{(1)}(\lambda x)\,d\lambda\,dt + \int_0^\infty \frac{e^{iut}x}{t^2-x^2}\,dt\ .$$

It follows that the solution of (3.10) is

$$(3.12)\qquad \phi(t) = -\frac{2}{\pi^2}\int_0^\infty h(x)\left\{-\frac{\pi i x}{2}\int_0^\alpha \lambda J_0(\lambda t)H_0^{(1)}(\lambda x)\,d\lambda + \frac{x}{t^2-x^2}\right\}dx$$

$$+ c_1 J_0(\alpha t)$$

or, by integrating the product of Bessel functions according to (2.12),

$$(3.13)\qquad \phi(t) = \frac{i\alpha}{\pi}\int_0^\infty h(x)\,\frac{[x^2 J_0(\alpha t)H_1^{(1)}(\alpha x) - xtJ_1(\alpha t)H_0^{(1)}(\alpha x)]}{x^2-t^2}\,dx$$

$$+ c_1 J_0(\alpha t)\ .$$

The equation

$$(3.14)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = h(x)$$

is the one we obtain if we replace $\alpha$ in (2.1) by $-|\alpha|$ . For this case, the transform of the solution, under the assumptions given above, must satisfy

$$2\int_\alpha^\infty \frac{\Phi(u)du}{u^2-\xi^2} = -\frac{H(\xi)}{\xi} .$$

The solution of this, as we can see from (3.9), is

$$(3.15) \qquad \Phi(u) = \frac{2u}{\pi^2\sqrt{u^2-\alpha^2}} \int_\alpha^\infty \frac{\sqrt{\xi^2-\alpha^2}\; H(\xi)d\xi}{\xi^2-u^2} + \frac{cu}{\sqrt{u^2-\alpha^2}} .$$

In accordance with our assumption that $\Phi(u) \to 0$ as $|u| \to \infty$, we take $c = 0$, and so

$$(3.16) \quad \Phi(u) = \frac{2u}{\pi^2\sqrt{u^2-\alpha^2}} \int_\alpha^\infty \frac{\sqrt{\xi^2-\alpha^2}\; H(\xi)d\xi}{\xi^2-u^2}$$

$$= \frac{2}{\pi^2}\int_0^\infty h(x)\, \frac{u}{\sqrt{u^2-\alpha^2}} \int_\alpha^\infty \frac{e^{ix\xi}\sqrt{\xi^2-\alpha^2}}{\xi^2-u^2}\, d\xi dx .$$

This can be inverted by using the same method that was used to invert (3.11). We note that

$$\frac{d}{du}\int_0^\alpha \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = -\frac{u\sqrt{\xi^2-\alpha^2}}{\sqrt{u^2-\alpha^2}\,(\xi^2-u^2)} + \frac{\xi}{\xi^2-u^2}$$

and

$$\frac{d}{du}\int_0^\xi \frac{\lambda d\lambda}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}} = \frac{\xi}{\xi^2-u^2} .$$

Then

$$\frac{u}{\sqrt{u^2-\alpha^2}}\int_\alpha^\infty \frac{e^{ix\xi}\sqrt{\xi^2-\alpha^2}\, d\xi}{\xi^2-u^2} = -\frac{d}{du}\int_\alpha^\infty e^{ix\xi}\int_0^\alpha \frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+\int_\alpha^\infty \frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{d}{du}\int_0^\alpha \frac{\lambda}{\sqrt{u^2-\lambda^2}} \int_\lambda^\infty \frac{e^{ix\xi}d\xi d\lambda}{\sqrt{\xi^2-\lambda^2}} + \frac{d}{du}\int_0^\alpha e^{ix\xi}\int_0^\xi \frac{\lambda d\lambda d\xi}{\sqrt{u^2-\lambda^2}\sqrt{\xi^2-\lambda^2}}$$

$$+ \int_\alpha^\infty \frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{\pi i}{2}\frac{d}{du}\int_0^\alpha \frac{\lambda H_0^{(1)}(\lambda x)d\lambda}{\sqrt{u^2-\lambda^2}} + \int_0^\infty \frac{\xi e^{ix\xi}d\xi}{\xi^2-u^2}$$

$$= -\frac{\pi i}{2}\int_0^\infty e^{iut}\, t\int_0^\alpha \lambda J_0(\lambda t)H_0^{(1)}(\lambda x)d\lambda dt + \int_0^\infty \frac{e^{iut}tdt}{t^2-x^2} .$$

Hence we can see that

$$(3.17)\quad \Phi(t) = \frac{2}{\pi^2}\int_0^\infty h(x)\left\{-\frac{\pi it}{2}\int_0^\alpha \lambda J_0(\lambda t)H_0^{(1)}(\lambda t)d\lambda + \frac{t}{t^2-x^2}\right\}dx$$

or

$$(3.18)\quad \Phi(t) = -\frac{i\alpha}{\pi}\int_0^\infty h(x)\frac{[xtJ_0(\alpha t)H_1^{(1)}(\alpha x)-t^2J_1(\alpha t)H_0^{(1)}(\alpha x)]dx}{x^2-t^2}$$

is a solution of (3.14).

The result (3.18) implies that the homogeneous equation

$$(3.19)\qquad \int_0^\infty \Phi(t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0$$

does not possess a non-trivial solution which satisfies the hypotheses of this section. Is there a solution which does not satisfy those hypotheses? This question prompts us to reexamine the possibility of retaining the term

(3.20) $$\Phi_0(u) = \frac{cu}{\sqrt{u^2-\alpha^2}}$$

in (3.15). If we retain this term we are led to a consideration of a generalized function as a possible solution of (3.19). This is so because

$$\Phi_0(u) = \frac{cu}{\sqrt{u^2-\alpha^2}} = c\left[\frac{u}{\sqrt{u^2-\alpha^2}} -1\right] + c$$

and inversion gives

$$\phi_0(t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-iut}\left[\frac{u}{\sqrt{u^2-\alpha^2}} - 1\right] du + \frac{c}{2\pi}\int_{-\infty}^{\infty} e^{-iut} du$$

$$= \frac{c}{\pi i}\int_{-\alpha}^{\alpha} \frac{ue^{-iut}du}{\sqrt{\alpha^2-u^2}} + c\delta(t)$$

(3.21) $$\phi_0(t) = - c\,[\alpha J_1(\alpha t) - \delta(t)]\ .$$

It can be verified, by substituting $\phi_0(t)$ in (3.19), that (3.21) does in fact satisfy the homogeneous equation (3.19).

It is interesting to observe that solutions of

$$\int_0^{\infty} \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} - c\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x)+\lambda\phi(x)$$

can be found by differentiating and integrating solutions of

$$\int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + c\, \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \quad .$$

For example, take

$$\int_0^\infty \psi_c(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0 \quad . \tag{3.22}$$

Assume that an analytic solution exists and deform the path of integration so that it lies above the real axis in the neighborhood of $x$ . Thus

$$-\pi i\, \psi_0(x) + \int_{\frown} \psi_0(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

and differentiation with respect to $x$ gives

$$-\pi i\, \psi_0'(x) \int_{\frown} \psi_0(t) \frac{d}{dx} \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} + \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0 \ .$$

This is the same as

$$-\pi i\, \psi_0'(x) + \int_{\frown} \psi_0(t) \frac{d}{dt} \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

and after an integration by parts it becomes.

$$-\pi i\, \psi_0'(x) - \psi_0(0) \frac{[e^{i|\alpha|x} - e^{-i|\alpha|x}]}{x}$$

$$- \int_{\frown} \psi_0'(t) \left\{ \frac{e^{i|\alpha|(x+t)}}{x+t} - \frac{e^{-i|\alpha|(x-t)}}{x-t} \right\} dt = 0$$

if there is no contribution from infinity. If we move the path back into coincidence with the real axis we find

$$(3.23) \quad -\psi_0(0)\,\frac{[e^{i|\alpha|x}-e^{-i|\alpha|x}]}{x}$$

$$-\int_{\frown} \psi_0'(t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt=0$$

Now we know that $\psi_0(t) = J_0(\alpha t)$ satisfies (3.22). Substituting this in (3.23) gives

$$-\frac{[e^{i|\alpha|x}-e^{-i|\alpha|x}]}{x}+\alpha\int_0^\infty J_1(\alpha t)\left\{\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0$$

which can be written

$$\int_0^\infty [\alpha J_1(\alpha t)-\delta(t)]\left\{\frac{e^{i|\alpha|(x+t)}}{x+t}-\frac{e^{-i|\alpha|(x-t)}}{x-t}\right\}dt = 0 \quad .$$

This again justifies the retention of the term (3.20).

4. <u>Reduction to a Hilbert-Riemann Boundary Value Problem.</u>

The equation

$$(4.1) \quad \int_0^\infty \phi(t) \left\{ \frac{e^{i\alpha(x+t)}}{x+t} + \nu \frac{e^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad x > 0 ,$$

where $\nu$ is plus or minus one, is a particular case of the more general equation

$$(4.2) \qquad \int_0^\infty [K(x-t) \pm K_1(x+t)]\, \phi(t)dt = h(x) + \lambda\phi(x) \qquad x > 0$$

where we are given $K_1(T) = K(-T)$ , $T > 0$ , and we define $K_1(T)$ for negative values of $T$ so that $K_1(T) = K(-T)$ holds for all real $T$ . We proceed to show that if the transform $\underline{K}(u)$ exists almost everywhere for $u$ real, then (4.2) can be reduced to the solution of the Hilbert-Riemann problem defined below.

If the Fourier transform of $K(T)$ is

$$\underline{K}(u) = \int_{-\infty}^\infty e^{iuT}K(T)dT ,$$

the Fourier transform of $K_1(T)$ is

$$\underline{K}_1(u) = \int_{-\infty}^\infty e^{iuT}K_1(T) = \int_{-\infty}^\infty e^{iuT}K(-T)dT = \int_{-\infty}^\infty e^{-iuT}K(T)dT = \underline{K}(-u)$$

Let the right hand transform of $\phi(t)$ be

$$\underline{\Phi}(\zeta) = \int_0^\infty e^{i\zeta t}\phi(t)dt \qquad \Im\zeta > 0$$

and let

$$\underline{\Phi}(\xi) = \lim_{n \to 0+} \underline{\Phi}(\xi + in) = \lim_{n \to 0+} \underline{\Phi}(\zeta)$$

$$\underline{\Phi}(-\xi) = \lim_{n \to 0-} \underline{\Phi}(-\xi - in) = \lim_{n \to 0+} \underline{\Phi}(-\xi + in)$$

$$= \lim_{n \to 0+} \underline{\Phi}(-\bar{\zeta}) = \underline{\Phi}(\xi e^{i\pi}) .$$

From the standpoint of generalized functions we have

$$(4.3) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} [\underline{K}(u)\ \underline{\Phi}(u) \pm \underline{K}(-u)\ \underline{\Phi}(-u)]du = h(x) + \lambda\phi(x) .$$

If we multiply this by $e^{i\zeta x}, \Im\zeta > 0$, and integrate from zero to infinity we get

$$(4.4) \qquad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\underline{K}(u)\ \underline{\Phi}(u) \pm \underline{K}(-u)\ \underline{\Phi}(-u)]}{u - \zeta} du = H(\zeta) + \lambda\ \underline{\Phi}(\zeta)$$

$$\Im\zeta > 0 .$$

Also if we multiply (4.3) by $e^{-i\zeta x}$, $\Im\zeta < 0$, and integrate from zero to infinity we get

$$(4.5) \qquad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{[\underline{K}(u)\ \underline{\Phi}(u) \pm \underline{K}(-u)\ \underline{\Phi}(-u)]}{u + \zeta} du = H(-\zeta) + \lambda\underline{\Phi}(-\zeta)$$

$$\Im\zeta < 0$$

which is the same as

$$(4.6) \quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\underline{\overline{K}}(-u)\ \underline{\overline{\Phi}}\ (-u) \pm \underline{\overline{K}}(u)\ \underline{\overline{\Phi}}\ (u)]}{u - \zeta}\, du = H(-\zeta) + \lambda\ \underline{\overline{\Phi}}\ (-\zeta)$$

$$\mathcal{I}\zeta < 0$$

and we can write this in the form

$$(4.7) \quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[\underline{\overline{K}}(-u)\ \underline{\overline{\Phi}}\ (-u) \pm \underline{\overline{K}}(u)\ \underline{\overline{\Phi}}\ (u)]}{u - \overline{\zeta}}\, du = H(-\overline{\zeta}) + \lambda\ \underline{\overline{\Phi}}\ (-\overline{\zeta})$$

$$\mathcal{I}\zeta > 0 .$$

If we add (4.7) to (4.4) when the upper sign is in force; and subtract (4.7) from (4.4) when the lower sign holds we find

$$(4.8) \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} [\underline{\overline{K}}(u)\ \underline{\overline{\Phi}}(u) \pm \underline{\overline{K}}(-u)\ \underline{\overline{\Phi}}(-u)]\left(\frac{1}{u-\zeta} - \frac{1}{u-\overline{\zeta}}\right) du$$

$$= H(\zeta) \pm H(-\overline{\zeta}) + \lambda[\ \underline{\overline{\Phi}}(\zeta) \pm \underline{\overline{\Phi}}(-\overline{\zeta})] .$$

$$\mathcal{I}\zeta > 0 .$$

Let us take the limit of (4.8) as $\zeta \to |\xi|$ ; and under the supposition that on any finite interval

$$\underline{\overline{K}}(u)\ \underline{\overline{\Phi}}(u) \pm \underline{\overline{K}}(-u)\ \underline{\overline{\Phi}}(-u)$$

is integrable in the Riemann sense or the Cauchy-Riemann sense if

there is a simple pole on the path of integration. The result is that

$$(4.9) \quad \underline{K}(\xi)\,\underline{\Phi}(\xi) \pm \underline{K}(-\xi)\,\underline{\Phi}(-\xi) = H(\xi) \pm H(-\xi) + \lambda[\underline{\Phi}(\xi) \pm \underline{\Phi}(-\xi)]$$

or

$$(4.10) \quad [\underline{K}(\xi)-\lambda]\,\underline{\Phi}(\xi) \pm [\underline{K}(-\xi)-\lambda]\,\underline{\Phi}(\xi e^{i\pi}) = H(\xi) \pm H(\xi e^{i\pi})$$

must hold for almost all values of $\xi \geq 0$. With respect to the z-plane

$$\xi^2 = z = x + iy$$

and with the notation

$$\underline{\Phi}(\sqrt{x}) = F(x) \ ; \ \underline{K}(\sqrt{x}) = Q(x) \ ; \ H(\sqrt{x}) = I(x) \ ;$$

the equation (4.10) is

$$(4.11) \quad [\,Q(x)-\lambda]F(x) \pm [\,Q\,(xe^{2\pi i})-\lambda]F(xe^{2\pi i}) = I(x) \pm I(xe^{2\pi i})$$

where $x \geq 0$. Equation (4.11) is the barrier equation for a Hilbert-Riemann problem. The problem is to find a function $F(z)$ analytic in the finite z-plane cut along the positive real axis and such that (4.11) is satisfied. Any singularities that we may be willing to admit for $F(z)$ at infinity or at the banks of the cut must of course be consistent with the existence of the integral (4.4) and must allow a reversal of the transformation of (4.2) into (4.4).

We have shown how the solution of (4.2) can be reduced to the

solution of the problem posed by the barrier equation (4.11). Since (4.11) can be solved by a well-known procedure, given for example in [5], we do not repeat it here.

## 5. Another Generalization

The equation

$$(5.1)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} + \nu\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x) + \lambda\phi(x) \quad x > 0,$$

where $\nu = \pm 1$, can also be regarded as a particular case of the equation

$$(5.2)\qquad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t} + k\,\frac{e^{-i\alpha(x-t)}}{x-t}\right\} dt = h(x) + \lambda\phi(x) \quad x > 0,$$

where $k$ is an arbitrary constant. The solution of (5.2) can be reduced to the solution of a Hilbert-Riemann problem by using the methods of Sections 3 and 4. We outline the reduction below.

We assume that $\phi(t)$ in (5.2) and its transform

$$\underline{\Phi}(\zeta) = \int_0^\infty e^{i\zeta t}\phi(t)dt$$

satisfy the conditions set in Section 3. If we take $\alpha > 0$ for convenience, then we find, just as in Section 3, that $\underline{\Phi}(u)$ must satisfy

$$(5.3)\qquad \int_\alpha^\infty \underline{\Phi}(u)\left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du = H(\xi) + \lambda\,\underline{\Phi}(\xi) \qquad \underline{0 < \alpha < \xi}$$

or

$$(5.4)\qquad \int_1^\infty \underline{\Phi}_1(u)\left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du = H_1(\xi) + \lambda\,\underline{\Phi}_1(\xi) \qquad 1 < \xi$$

where

$$\Phi_1(u) = \Phi(\alpha u) \; ; \; H_1(\xi) = H(\alpha\xi) \; .$$

Equation (5.4) is an equation of Wiener-Hopf type which can be solved by using Mellin transforms.

We assume that the Mellin transform

$$(5.5) \qquad \Psi(s) = \int_1^\infty u^{s-1} \Phi_1(u)du$$

exists for all $s$ such that $Rs = a$ ; $0 < a < 1$ and that

$$(5.6) \qquad \Phi_1(u) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} u^{-s}\Psi(s)ds \; .$$

the transform for $\Psi(s)$ is analytic in the left half plane $\mathcal{R}s < a$ and $\Psi(s) \to 0$ as $|s-a| \to \infty$ , $\frac{\pi}{2} < \arg(s-a) < \frac{3\pi}{2}$ . The substitution of (5.6) in (5.4) gives

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Psi(s) \int_0^\infty u^{-s} \left[ \frac{1}{u+\xi} + \frac{k}{u-\xi} \right] du ds = H_1(\xi) + \lambda\Phi_1(\xi)$$

or

$$(5.7) \qquad \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \xi^{-s} \left[ \frac{1 + k \cos \pi s}{\sin \pi s} \right] \Psi(s)ds = H_1(\xi) + \lambda \Phi_1(\xi) \; .$$

The result of multiplying (5.7) by $\xi^{z-1}$ , $\mathcal{R} z < a$ , and integrating from one to infinity is

$$(5.8) \qquad \frac{1}{2i}\int_{a-i\infty}^{a+i\infty} \frac{1}{s-z} \left[ \frac{1 + k \cos \pi s}{\sin \pi s} \right] \Psi(s)ds = \mathcal{H}(z) + \lambda\Psi(z)$$

$$Rz < a$$

where

$$\mathcal{H}(z) = \int_1^\infty \xi^{z-1} H_1(\xi) d\xi$$

is supposed to exist for $Rz = a$ .

The path of integration in (5.8) is parallel to the imaginary axis in the s-plane and we denote the path of $\Gamma$ . Let $\sigma$ be a point on $\Gamma$ and let $z \to \sigma$ . The limit process yields

$$(5.9)\quad \frac{\pi}{2}\left[\frac{1 + k\cos\pi\sigma}{\sin\pi\sigma}\right]\Psi(\sigma) + \frac{1}{2i}\int_\Gamma \frac{1}{s-\sigma}\left[\frac{1+k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds$$

$$= \mathcal{H}(\sigma) + \lambda\Psi(\sigma)$$

or

$$(5.10)\quad \pi\left[\frac{1+k\cos\pi\sigma}{\sin\pi\sigma}\right]\Psi(\sigma)+\lim_{z\to\sigma+0}\frac{1}{2i}\int_\Gamma\frac{1}{s-z}\left[\frac{1+k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds$$

$$= \mathcal{H}(\sigma) + \lambda\Psi(\sigma) .$$

That is, if $F(z)$ is the sectionally analytic function defined by

$$F(z) = \begin{cases} \Psi(z) & \text{analytic for } Rz < a \\ \dfrac{1}{2i}\displaystyle\int_\Gamma \frac{1}{s-z}\left[\frac{1+k\cos\pi s}{\sin\pi s}\right]\Psi(s)ds & \text{analytic for } Rz > a , \end{cases}$$

then $F(z)$ must satisfy the Hilbert-Riemann problem posed by the barrier equation

$$(5.12) \qquad \left[ \frac{\pi\,(1 + k \cos \pi\sigma)}{\sin \pi\sigma} - \lambda \right] F(\sigma - 0) + F(\sigma + 0) = \mathcal{H}(\sigma)$$

which can be solved by known methods. Once $F(z)$ is known, $\Psi(s)$ can be found from (5.11) and it follows by inversion that the solution of (5.2) is

$$\Phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itu}\,\frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \left(\frac{u}{\alpha}\right)^{-s} \Psi(s)\,ds\,du\ .$$

## 6. A Related Equation

In this section we show that the solution of the equation

$$(6.1) \qquad \int_0^\infty \phi(t) \left\{ \frac{e^{-i\alpha(x+t)}}{x+t} + \frac{ke^{-i\alpha(x-t)}}{x-t} \right\} dt = h(x) + \lambda\phi(x) \qquad x > 0$$

can be reduced to the solution of an equation already considered.

We assume that $\phi(t)$ and

$$\underline{\Phi}(\zeta) = \int_0^\infty e^{i\zeta t}\, \phi(t)dt$$

satisfy the conditions prescribed in Section 3. By the method of that section we are led to

$$(6.2) \qquad \frac{1}{2} \int_{-\infty}^{\infty} \frac{\underline{\Phi}(u)\ \mathrm{sgn}\ (u+\alpha)}{u+\zeta}\, du + \frac{k}{2} \int_{-\infty}^{\infty} \frac{\underline{\Phi}(u)\ \mathrm{sgn}\ (u-\alpha)du}{u-\zeta}$$

$$= H(\zeta) + \lambda\, \underline{\Phi}(\zeta) \qquad \mathcal{I}\zeta > 0 .$$

Suppose that $\alpha > 0$ . The transform $\underline{\Phi}(\zeta)$ is analytic in $\mathcal{I}\zeta > 0$ and $\underline{\Phi}(\zeta) \longrightarrow 0$ as $|\zeta| \longrightarrow \infty$ , $0 \arg \zeta < \pi$ . Hence an application of the Cauchy integral formula to (6.2) produces

$$(6.3) \qquad - \int_{-\pi}^{-\alpha} \frac{\underline{\Phi}(u)du}{u+\zeta} + k \int_{\alpha}^{\infty} \frac{\underline{\Phi}(u)du}{u-\zeta} - k\pi i\, \underline{\Phi}(\zeta) = H(\zeta) + \lambda\, \underline{\Phi}(\zeta)$$

or

$$(6.4) \qquad \int_\alpha^\infty \frac{[\Phi(ue^{i\pi}) + k\,\Phi(u)]}{u-\zeta}\,du - k\pi i\,\Phi(\zeta) = H(\zeta) + \lambda\,\Phi(\zeta)$$

$$\mathscr{I}\zeta > 0 .$$

If we let $\zeta$ in (6.4) approach $\xi$, $\xi > \alpha$, and then let $\zeta$ approach $\xi e^{i\pi}$ we obtain

$$(6.5) \qquad \int_\alpha^\infty \frac{[\Phi(ue^{i\pi}) + k\,\Phi(u)]}{u-\xi}\,du + \pi i\,\Phi(\xi e^{i\pi}) = H(\xi) + \lambda\,\Phi(\xi)$$

$$(6.6) \qquad \int_\alpha^\infty \frac{[\Phi(ue^{i\pi}) + k\,\Phi(u)]}{u+\xi}\,du - k\pi i\,\Phi(\xi e^{i\pi}) = H(\xi e^{i\pi})$$

$$+ \lambda\,\Phi(\xi e^{i\pi}) .$$

If we multiply equation (6.5) by $k$ and add the resulting equation to (6.6) we get

$$(6.7) \qquad k\int_\alpha^\infty \frac{[\Phi(ue^{i\pi}) + k\,\Phi(u)]}{u-\xi}\,du + \int_\alpha^\infty \frac{[\Phi(ue^{i\pi}) + k\,\Phi(u)]}{u+\xi}\,du$$

$$= kH(\xi) + H(\xi e^{i\pi}) + \lambda[k\,\Phi(\xi) + \Phi(\xi e^{i\pi})] .$$

If we introduce

$$(6.8) \qquad k\,\Phi(u) + \Phi(ue^{i\pi}) = \Omega(u)$$

equation (6.7) becomes

$$(6.9) \qquad \int_{\alpha}^{\infty} \Omega(u) \left[\frac{1}{u+\xi} + \frac{k}{u-\xi}\right] du$$

$$= kH(\xi) + H(\xi e^{i\pi}) + \lambda\Omega(\xi) .$$

This equation was discussed in Section 5 and it was shown there how it could be solved by reducing it to a barrier equation. Once $\Omega(u)$ has been found, the transform $\underline{\Phi}(u)$ can be determined by solving the Hilbert-Riemann problem posed by the barrier equation (6.8).

# R E F E R E N C E S

[1] Jones, D. S.,
<u>Diffraction at High Frequencies by a Circular Disc</u>, Research Report No. EM-188, New York University, Courant Institute of Mathematical Sciences, July 1963.

[2] Jones, D. S.,
<u>On a Certain Singular Integral Equation I</u>, J. of Math. and Physics, Vol. XLIII, No. 1, March 1964.

[3] Jones, D. S.,
<u>On a Certain Singular Integral Equation II</u>, J. of Math. and Physics, Vol. XLIII, No. 3, Sept. 1964.

[4] Peters, A. S.,
<u>Certain Dual Integral Equations and Sonine's Integrals</u>, Research Report No. IMM-285, New York University, Courant Institute of Mathematical Sciences, August 1961.

[5] Muskhelishvili, N. I.
<u>Singular Integral Equations</u>, P. Noordhoff N. V., Groningen, Holland (1953).

Security Classification

## DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

| | |
|---|---|
| 1. ORIGINATING ACTIVITY (Corporate author)<br>Courant Institute of Mathematical Sciences<br>New York University | 2a. REPORT SECURITY CLASSIFICATION<br>Not classified<br>2b. GROUP None |

3. REPORT TITLE

An Integral Equation from Diffraction Theory

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report February 1965

5. AUTHOR(S) (Last name, first name, initial)

Peters, Arthur S.

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| February 1965 | 38 | 5 |

| | |
|---|---|
| 8a. CONTRACT OR GRANT NO.<br>Nonr-285(55)<br>b. PROJECT NO.<br>NR 062-160<br>c.<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>IMM-NYU 337<br>9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)<br>None |

10. AVAILABILITY/LIMITATION NOTICES

All distribution to be controlled

| | |
|---|---|
| 11. SUPPLEMENTARY NOTES<br>None | 12. SPONSORING MILITARY ACTIVITY<br>U.S. Navy, Office of Naval Research<br>207 West 24th St., New York, N. Y. |

13. ABSTRACT


This report presents several methods for solving

$$(A)\quad \int_0^\infty \phi(t)\left\{\frac{e^{i\alpha(x+t)}}{x+t}+\frac{ke^{i\beta(x-t)}}{x-t}\right\}dt = h(x)+\lambda\phi(x) \qquad 0 < x$$

where k; $\lambda$ are arbitrary constants; $\alpha$ is real and $\beta = \pm\alpha$. Equation (A) is a generalization of an equation which arises in diffraction theory.

The more general equation

$$(B)\quad \int_0^\infty K(x-t)\phi(t)dt + \int_0^\infty K_1(x+t)\phi(t)dt = h(x)+\lambda\phi(x) \qquad 0 < x$$

is also studied; and it is shown how the solution of (B) can be reduced to the solution of a Hilbert-Riemann problem when $K(\tau)$ is related to $K_1(\tau)$ in a special way.


DD FORM 1 JAN 64 1473

Security Classification

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| | | | | | | |

## INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

# Approved Distribution List for Nonr 285(55)

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Att: Code 438 (3)

Commanding Officer
Office of Naval Research
Branch Office
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois (1)

Commanding Officer
Office of Naval Research
Branch Office
207 West 24th Street
New York 11, New York (1)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California (1)

Commanding Officer
Office of Naval Research
Branch Office
1000 Geary Street
San Francisco 24, California (1)

Commanding Officer
Office of Naval Research
Navy No. 100, F. P. O.
New York, New York (5)

Director
Naval Research Laboratory
Washington 25, D. C.
Att: Code 2021 (6)

Defense Documentation Center
Cameron Station
Alexandria, Virginia 22314 (10)

Professor W. R. Sears
Director
Grad. Sch. of Aeronautical Engrg.
Cornell University
Ithaca, New York (1)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D. C.
Att: Research Division (1)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D. C.
Att: Research and Dev. Div. (1)

Office of Ordnance Research
Department of the Army
Washington 25, D. C. (1)

Commander
Air Res. and Development Command
P. O. Box 1395
Baltimore 18, Maryland
Att: Fluid Mechanics Div. (1)

Director of Research
National Advisory Comm. for Aeron.
1724 F Street, N. W.
Washington 25, D. C. (1)

Director
Langley Aeronautical Laboratory
National Advisory Comm. for Aeron.
Langley Field, Virginia (1)

Director
National Bureau of Standards
Washington 25, D. C.
Att: Fluid Mechanics Section (1)

Professor Richard Courant
Courant Inst. of Math. Sciences
New York University
251 Mercer Street
New York, New York 10012 (1)

Professor G. Kuerti
Dept. of Mechanical Engineering
Case Institute of Technology
Cleveland, Ohio (1)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D. C.
Att: Research Division (1)
Code 420 Prelim. Design (1)

Commander
Naval Ordnance Test Station
3202 E. Foothill Blvd.
Pasadena, California (1)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D. C.
Att: Hydromechanics Lab. (1)
Hydrodynamics Div. (1)
Library (1)
Ship Division (1)

California Inst. of Technology
Hydrodynamics Laboratory
Pasadena 4, California (1)

Professor A. T. Ippen
Hydrodynamics Laboratory
Massachusetts Inst. of Technology
Cambridge 39, Massachusetts (1)

Dr. Hunter Rouse, Director
Iowa Inst. of Hydraulic Research
State University of Iowa
Iowa City, Iowa (1)

Stevens Institute of Technology
Experimental Towing Tank
711 Hudson Street
Hoboken, New Jersey (1)

Dr. L. G. Straub
St. Anthony Falls Hydraulic Lab.
University of Minnesota
Minneapolis 14, Minnesota (1)

Dr. G. H. Hickox
Engineering Experiment Station
University of Tennessee
Knowville, Tennessee (1)

Chief of Naval Research
Department of the Navy
Washington 25, D. C.
Att: Code 416 (1)
Code 460 (1)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D. C.
Att: Research Division (1)

Hydrographer
Department of the Navy
Washington 25, D. C. (1)

Director
Waterways Experiment Station
Box 631
Vicksburg, Mississippi (1)

Office of the Chief of Engineers
Department of the Army
Gravelly Point
Washington 25, D. C. (1)

Beach Erosion Board
U. S. Army Corps of Engineers
Washington 25, D. C. (1)

Commissioner
Bureau of Reclamation
Washington 25, D. C. (1)

Dr. G. H. Keulegan
National Hydraulic Laboratory
National Bureau of Standards
Washington 25, D. C. (1)

Brown University
Graduate Div. of Applied Math.
Providence 12, Rhode Island (1)

California Inst. of Technology
Hydrodynamics Laboratory
Pasadena 4, California
Att: Professor M. S. Plesset (1)
Professor V. A. Vanoni (1)

Mr. C. A. Gongwer
Aerojet General Corporation
6352 N. Irwindale Avenue
Asusa, California (1)

Professor M. L. Albertson
Department of Civil Engineering
Colorado A. and M. College
Fort Collins, Colorado (1)

Professor G. Birkhoff
Department of Mathematics
Harvard University
Cambridge 38, Massachusetts (1)

Massachusetts Inst. of Technology
Department of Naval Architecture
Cambridge 39, Massachusetts (1)

Dr. R. R. Revelle
Scripps Inst. of Oceanography
La Jolla, California (1)

Stanford University
Applied Math. and Stat. Laboratory
Stanford, California (1)

Professor J. W. Johnson
Fluid Mechanics Laboratory
University of California
Berkeley 4, California (1)

Professor H. A. Einstein
Department of Engineering
University of California
Berkeley 4, California (1)

Dean K. L. Schoenherr
College of Engineering
University of Notre Dame
Notre Dame, Indiana (1)

Director
Woods Hole Oceanographic Institute
Woods Hole, Massachusetts (1)

Hydraulics Laboratory
Michigan State College
East Lansing, Michigan
Att: Professor R. R. Henry (1)